We will follow the same procedure to validate the equation $\Pr\{\gamma_t^{\mathrm{D}} \le x\} = G(x)$ for all x, where γ_t^{D} is the excess life in the delayed (stationary) renewal process. Let

$$A_x^{\mathrm{D}}(t) = \Pr\{\gamma_t^{\mathrm{D}} > x\}, \qquad \text{and} \qquad A_x(t) = \Pr\{\gamma_t > x\},$$

where γ_t is the excess life in an ordinary renewal process. The standard renewal argument leads to
(as in 7.1)

$$A_x^{\mathrm{D}}(t) = 1 - G(t+x) + \int_0^t A_x(t-y)\, dG(y),$$

or

$$A_x^{\mathrm{D}}(t) = 1 - G(t+x) + G * A_x(t). \tag{7.5}$$

The renewal equation

$$A_x(t) = 1 - F(t+x) + F * A_x(t)$$

appeared earlier, in our deliberations of Section 6. By virtue of Theorem 4.1, the solution can be represented in the form

$$A_x(t) = a_x(t) + M * a_x(t), \tag{7.6}$$

where

$$a_x(t) = 1 - F(t+x).$$

Inserting (7.6) into (7.5) and citing the formula $M_{\mathrm{D}}(t) = G(t) + G * M(t)$ (this comes out directly from the definitions involved), we obtain

$$\begin{aligned} A_x^{\mathrm{D}}(t) &= 1 - G(t+x) + G * a_x(t) + G * M * a_x(t) \\ &= 1 - G(t+x) + M_{\mathrm{D}} * a_x(t) \\ &= 1 - G(t+x) + \int_0^t a_x(t-y)\, dM_{\mathrm{D}}(y). \end{aligned}$$

(similar to 7.6)

Now $a_x(t-y) = 1 - F(t+x-y)$ and $M_{\mathrm{D}}(y) \equiv y/\mu$, so that $dM_{\mathrm{D}}(y) = \mu^{-1}\, dy$. Then

$$\begin{aligned} A_x^{\mathrm{D}}(t) &= 1 - G(t+x) + \mu^{-1} \int_0^t \{1 - F(t+x-y)\}\, dy \\ &= 1 - G(t+x) + \mu^{-1} \int_x^{t+x} \{1 - F(u)\}\, du \\ &= 1 - G(t+x) + G(t+x) - G(x) \\ &= 1 - G(x), \end{aligned}$$

as was to be shown.

C. CUMULATIVE AND RELATED PROCESSES

Suppose associated with the ith unit or lifetime interval is a second random variable Y_i ($\{Y_i\}$ identically distributed) in addition to the lifetime X_i. We allow X_i and Y_i to be dependent, but assume that the pairs $(X_1, Y_1), (X_2, Y_2), \ldots$ are independent. We use the notation $F(x) = \Pr\{X_i \le x\}$, $G(y) = \Pr(Y_i \le y\}$, $\mu = E[X_i]$, and $\nu = E[Y_i]$.

A number of problems of practical and theoretical interest have a natural formulation in these terms.

I. *Renewal Processes Involving Two Components to Each Renewal Interval*

Suppose that

$$Y_i \text{ represents a portion of the duration } X_i.$$

Figure 7 illustrates the model. In Fig. 7 we have depicted the Y portion occurring at the beginning of the interval, but this is not essential for the results that follow.

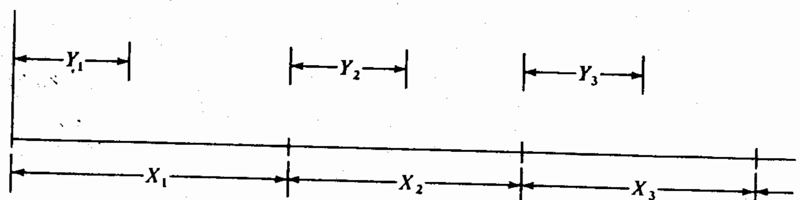

FIG. 7. A renewal process in which an associated random variable Y_i represents a portion of the ith renewal interval.

Let $p(t)$ be the probability that t falls in a Y portion of some renewal interval. By conditioning on the length of the first interval $X_1 = x$ and distinguishing the two possibilities $x < t$ and $x \ge t$, we arrive (by what is now a routine methodology) at the renewal equation

$$p(t) = \Pr\{t \text{ is covered by } Y_1\} + \int_0^t p(t-\xi)\, dF(\xi).$$

Let

$$I_{Y_1}(t) = \begin{cases} 1, & \text{if } Y_1 \text{ covers } t, \\ 0, & \text{if } Y_1 \text{ does not cover } t. \end{cases}$$

Then

$$\Pr\{t \text{ is covered by } Y_1\} = E[I_{Y_1}(t)],$$

and

$$\int_0^\infty \Pr\{t \text{ is covered by } Y_1\}\,dt = \int_0^\infty E[I_{Y_1}(t)]\,dt$$
$$= E\left[\int_0^\infty I_{Y_1}(t)\,dt\right]$$
$$= E[Y_1] = \nu,$$

since the totality of points covered by Y_1 is Y_1. Now applying the renewal theorem, we conclude that if F is nonarithmetic and $\Pr\{t$ is covered by $Y_1\}$ is directly Riemann integrable, then

$$\lim_{t\to\infty} p(t) = \mu^{-1}\int_0^\infty \Pr\{t \text{ is covered by } Y_1\}\,dt$$
$$= \nu/\mu. \qquad (7.7)$$

a rather obvious result

Here are some concrete examples.

(a) *A Replacement Model*

Consider a replacement model in which replacement is not instantaneous. Let Y_i be the operating time and Z_i the lag period preceding installment of the $(i+1)$st operating unit. (The delay in replacement can be conceived as a period of repair of the service unit.) We assume that the sequence of times between successive replacements $X_k = Y_k + Z_k$, $k = 1, 2, \ldots$, constitutes a renewal process. Then $p(t)$, the probability that the system is in operation at time t, converges to $E[Y_1]/E[X_1]$, provided the distribution of X_k is nonarithmetic.

(b) *A Queuing Model*

If arrivals to a queue follow a Poisson process, then the successive times X_k from the commencement of the kth busy period to the start of the next busy period form a renewal process. (A busy period is an uninterrupted duration when the queue is not empty.) Each X_k is composed of a busy portion Z_k and an idle portion Y_k. Then $p(t)$, the probability that the queue is empty at time t, converges to $E[Y_1]/E[X_1]$.

(c) *A Counter Problem*

Let X_k, $k = 1, 2, \ldots$, denote the sequence of times between successive recorded particles in a counter and let Y_k represent the dead (blocked) time during the X_k renewal period. Then $p(t)$, the probability that the counter is blocked at time t, converges to $E[Y_1]/E[X_1]$.

II. *Cumulative Processes*

Interpret Y_i as a cost or value, etc., associated with the ith renewal cycle. A class of problems with natural setting in this general context of pairs (X_i, Y_i), where X_i generates a renewal process, will now be considered. Interest here focuses on the so-called *cumulative process*

$$W(t) = \sum_{k=1}^{N(t)+1} Y_k,$$

the accumulated costs or what-have-you up to time t (assuming transactions are made at the beginning of a renewal cycle). By conditioning on the time $X_1 = x$ until the first renewal, and examining the two possibilities $x > t$ and $x \le t$, we secure for $A(t) = E[W(t)]$ the renewal equation

$$A(t) = E[Y_1] + \int_0^t A(t-x)\,dF(x).$$

An appeal to Theorem 4.1 yields the formula

$$A(t) = E[Y_1] + \int_0^t E[Y_1]\,dM(x)$$
$$= E[Y_1][1 + M(t)].$$

It follows immediately that, where F is nonarithmetic and $h > 0$, then

$$\lim_{t\to\infty}[A(t) - A(t-h)] = E[Y_1]h/\mu,$$

and in any case,

$$\lim_{t\to\infty}\frac{1}{t}A(t) = E[Y_1]/\mu.$$

This justifies the interpretation of $E[Y_1]/\mu$ as a long-run mean cost, value, etc., per unit time, an interpretation that was used repeatedly in the examples of Section 3.

Here are some examples of cumulative processes.

(a) *Replacement Models*

Suppose Y_i is the cost of the ith replacement. Let us suppose that under an age-replacement strategy (see Example B, Section 3) a planned replacement at age T costs c_1 dollars, while a failure replaced at time $x < T$ costs c_2 dollars. If Y_k is the cost incurred at the kth replacement cycle, then

$$Y_k = \begin{cases} c_1 & \text{with probability } 1 - F(T), \\ c_2 & \text{with probability } F(T), \end{cases}$$

and $E[Y_k] = c_1[1 - F(T)] + c_2 F(T)$. Since the expected length of a replacement cycle is

$$E[\min\{X_k, T\}] = \int_0^T [1 - F(x)]\, dx,$$

we have that the long-run cost per unit time is

$$\frac{c_1[1 - F(T)] + c_2 F(T)}{\int_0^T [1 - F(x)]\, dx},$$

and in any particular situation a routine calculus exercise or recourse to numerical computation produces the value of T that minimizes the long-run cost per unit time.

Under a block replacement policy, there is one planned replacement every T units of time and, on the average, $M(T)$ failure replacements, so the expected cost is $E[Y_k] = c_1 + c_2 M(T)$, and the long-run mean cost per unit time is $\{c_1 + c_2 M(T)\}/T$.

(b) *Counter Models*

In a counter model (see Example C, Section 3), let Y_k be the number of unregistered signals that arise during the period X_k between the $(k-1)$st and kth recorded signals. Then the long-run mean number of uncounted particles per unit time is $E[Y_1]/E[X_1]$.

(c) *Risk Theory*

Suppose claims arrive at an insurance company according to a renewal process with interoccurrence times $X_1, X_2, \ldots$. Let Y_k be the magnitude of the kth claim. Then $W(t) = \sum_{k=0}^{N(t)+1} Y_k$ represents the cumulative amount claimed up to time t, and the long-run mean claim rate is

$$\lim_{t \to \infty} \frac{1}{t} E[W(t)] = E[Y_1]/E[X_1].$$

D. TERMINATING RENEWAL PROCESSES

Suppose we allow the possibility of infinite interoccurrence times in a renewal process. Such a process is called a *terminating* renewal process, since the renewals cease at the first infinite interoccurrence time. The situation is diagrammed in Fig. 8.

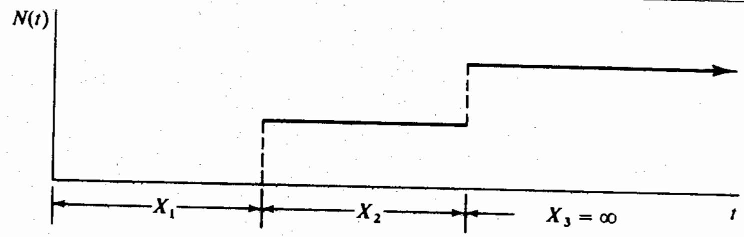

FIG. 8.

Let $L = F(\infty) = \Pr\{X_k < \infty\} < 1$ and $1 - L = \Pr\{X_k = \infty\} > 0$. Then the total number of renewals in all time, denoted by $N(\infty)$, is a finite-valued random variable and follows the geometric probability law

$$\Pr\{N(\infty) \geq k\} = L^k, \qquad k = 0, 1, 2, \ldots,$$

with

$$E[N(\infty)] = \sum_{k=1}^{\infty} \Pr\{N(\infty) \geq k\}$$
$$= L/(1 - L).$$

The realizations of the termination process still have the property $N(t) \geq k$ if and only if $S_k \leq t$, so that

$$\Pr\{N(t) \geq k\} = \Pr\{S_k \leq t\} = F_k(t),$$

and

$$M(t) = E[N(t)] = \sum_{k=1}^{\infty} F_k(t) < \sum_{k=1}^{\infty} L^k = L/(1 - L).$$

Moreover, the renewal argument continues to work, entailing the equation

$$M(t) = F(t) + \int_0^t M(t - x)\, dF(x).$$ though Theorem is all right

However, the renewal theorem is *not* automatically applicable owing to the fact that F is not a proper probability distribution function. Fortunately, there is often a way to overcome this lacuna. Suppose that

$$g(s) = \int_0^{\infty} e^{sx}\, dF(x)$$

is a finite function of s for $s \geq 0$. Then g will be continuous and $g(0) = L < 1$ and $\lim_{s \to \infty} g(s) = \infty$, implying the existence of a unique positive value $s_0 = \lambda > 0$, for which

$$g(\lambda) = \int_0^{\infty} e^{\lambda x}\, dF(x) = 1.$$