all sojourn times are assumed independent. The unconditional distribution function of the sojourn time in a state i is $F_i(t) = \sum_{j=1}^{r} P_{ij} F_{ij}(t)$, which is postulated to have a finite mean μ_i. Assume that the Markov chain is irreducible and recurrent, with stationary distribution given by $\pi_j = \sum_i \pi_i P_{ij}$.

Suppose the process starts in a fixed state i, and let a state k be prescribed. Call the duration between one visit to state i and the next an *i-cycle*. The sequence of times between these successive visits to state i forms a renewal process. From relation (7.7) and assuming at least one F_i is not arithmetic, the probability $p_k(t)$ of being in state k at time t converges to the mean time spent in state k during an i-cycle divided by the mean duration of an i-cycle. By the law of total probability, the mean time in state k during an i-cycle is the product of μ_k times the mean number of visits to k in the intervening time between successive visits to state i. The second factor depends only on the discrete-time Markov chain of state visits and therefore is necessarily proportional to π_k. It follows that

$$\lim_{t\to\infty} p_k(t) = c\pi_k \mu_k,$$

when c is a constant of proportionality. Since these probabilities necessarily sum to 1, $c = 1/(\pi_1\mu_1 + \cdots + \pi_r\mu_r)$.

F. CENTRAL LIMIT THEOREM FOR RENEWALS

Theorem 7.1. *Let $\{X_n\}$ be a renewal process for which $\mu = E[X_1] < \infty$ and $\sigma^2 = E[(X_1 - \mu)^2] < \infty$. Then*

$$\lim_{t\to\infty} \Pr\left\{\frac{N(t) - t/\mu}{\sqrt{t\sigma^2/\mu^3}} < x\right\} = \Phi(x),$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-\tfrac{1}{2}u^2)\, du$$

is the normal integral.

Proof. The proof rests on the central limit theorem for $S_n = X_1 + \cdots + X_n$ and the basic identity of realizations of the process in the form $\{N(t) < n\}$ if and only if $\{S_n > t\}$.

Let x be fixed and let $n \to \infty$ and $t \to \infty$ in such a way that

$$\lim_{\substack{t\to\infty\\ n\to\infty}} \frac{t - n\mu}{\sigma\sqrt{n}} = -x.$$

Then, by the usual central limit theorem,

$$\lim_{\substack{t\to\infty\\ n\to\infty}} \Pr\{S_n > t\} = \lim_{\substack{t\to\infty\\ n\to\infty}} \Pr\left\{\frac{S_n - n\mu}{\sigma\sqrt{n}} > -x\right\} = 1 - \Phi(-x) = \Phi(x).$$

But then

$$\begin{aligned}\Phi(x) &= \lim_{\substack{t\to\infty\\ n\to\infty}} \Pr\{S_n > t\}\\ &= \lim_{\substack{t\to\infty\\ n\to\infty}} \Pr\{N(t) < n\}\\ &= \lim_{\substack{t\to\infty\\ n\to\infty}} \Pr\left\{\frac{N(t) - t/\mu}{\sqrt{t\sigma^2/\mu^3}} < \frac{n - t/\mu}{\sqrt{t\sigma^2/\mu^3}}\right\}\\ &= \lim_{t\to\infty} \Pr\left\{\frac{N(t) - t/\mu}{\sqrt{t\sigma^2/\mu^3}} < x\right\},\end{aligned}$$

since $(n - t/\mu)/\sqrt{t\sigma^2/\mu^3} \to x$ as $t \to \infty$, $n \to \infty$ in such a manner that $(t - n\mu)/\sqrt{n\sigma^2} \to -x$. ∎

The preceding analysis was conducted in a formal manner and needs tightening. The student may try to supply the epsilonics.

G. RUIN IN RISK THEORY

Let $N(t)$ be the number of claims incurred by an insurance company over the time interval $(0, t]$. Assume $N(t)$ is a Poisson process with parameter λ. Assume, moreover, that the magnitudes of the successive claims $Y_1, Y_2, Y_3, \ldots$ are independent identically distributed random variables having distribution function $G(x)$. Let the inflow of cash (premiums, investments, etc.) be c dollars per unit time and suppose the initial capital of the company is z. Then at time t, the cash balance is

$$\Gamma(t) = z + ct - \sum_{i=1}^{N(t)} Y_i,$$

where Y_i is the magnitude of the ith successive claim. It is of interest to ascertain the probability of continual solvency as a function of z. That is, we wish to determine

$$R(z) = \Pr\left\{z + ct - \sum_{i=1}^{N(t)} Y_i > 0, \text{ for all } t\right\} \tag{7.8}$$

$$= \text{probability of no ruin with initial capital } z.$$

We apply the renewal argument conditioning on the time T_1 of the first Poisson event. Together with the law of total probabilities, we obtain

$$R(z) = \int_0^\infty \Pr\left\{z + ct - \sum_{i=1}^{N(t)} Y_i > 0 \text{ for all } t \mid T_1 = \tau\right\} \lambda e^{-\lambda\tau}\, d\tau. \tag{7.9}$$

But another conditioning on the value of Y_1 entails

$$\Pr\left\{z + ct - \sum_{i=1}^{N(t)} Y_i > 0 \text{ for all } t \mid T_1 = \tau\right\}$$
$$= \int_0^\infty \Pr\left\{z + ct - \sum_{i=1}^{N(t)} Y_i > 0 \text{ for all } t \mid Y_1 = y,\ T_1 = \tau\right\} dG(y). \tag{7.10}$$

The process $\Gamma(t)$ renews itself immediately after time τ holding the new initial capital $z + c\tau - y$, given $T_1 = \tau$, $Y_1 = y$. Therefore,

$$\Pr\left\{z + ct - \sum_{i=1}^{N(t)} Y_i > 0 \text{ for all } t \mid Y_1 = y,\ T_1 = \tau\right\} = R(z + c\tau - y). \tag{7.11}$$

Of course, $R(u) = 0$ for $u < 0$.

The facts of (7.10) and (7.11) implemented into (7.9) produce the integral equation

$$R(z) = \int_0^\infty \left(\int_0^{z+c\tau} R(z + c\tau - y)\, dG(y)\right) \lambda e^{-\lambda\tau}\, d\tau.$$

A change of variables $t = z + c\tau$ in the outer integral and rearrangement gives

$$R(z) e^{-\lambda z/c} = \frac{\lambda}{c} \int_z^\infty \left(\int_0^t R(t - y)\, dG(y)\right) e^{-\lambda t/c}\, dt.$$

The representation assures that $R(z)$ is differentiable, and differentiation yields

$$e^{-\lambda z/c}\left[R'(z) - \frac{\lambda}{c} R(z)\right] = -\frac{\lambda}{c} e^{-\lambda z/c} \int_0^z R(z - y)\, dG(y),$$

or, equivalently,

$$R'(z) = \frac{\lambda}{c} R(z) - \frac{\lambda}{c} \int_0^z R(z - y)\, dG(y).$$

Integrating both sides with respect to z gives

$$R(w) - R(0) = \frac{\lambda}{c} \int_0^w R(z)\, dz - \frac{\lambda}{c} \int_0^w \left(\int_0^z R(z - y)\, dG(y)\right) dz.$$

Interchanging the orders of integration and then a change of variable $\xi = z - y$ leads to

$$R(w) = R(0) + \frac{\lambda}{c} \int_0^w R(z)\, dz - \frac{\lambda}{c} \int_0^w \left(\int_0^{w-y} R(\xi)\, d\xi\right) dG(y).$$

Define $S(x) = \int_0^x R(\xi)\, d\xi$. Next perform an integration by parts to obtain

$$R(w) - R(0) = \frac{\lambda}{c} S(w) - \frac{\lambda}{c}\left\{S(w) - \int_0^w R(w - y)[1 - G(y)]\, dy\right\},$$

or

$$R(w) = R(0) + \frac{\lambda}{c} \int_0^w R(w - y)[1 - G(y)]\, dy.$$

These manipulations have produced a renewal equation with an improper density $(\lambda/c)[1 - G(y)]$, since

$$\int_0^\infty \frac{\lambda}{c}[1 - G(y)]\, dy = \frac{\lambda}{c} E[Y_1] = \frac{\lambda}{c}\mu.$$

→ unique bounded solution

If $\lambda\mu/c > 1$, it is certain that $R(z) = 0$ (why?). (Note that $\lambda\mu$ is the expected outflow per unit time servicing claims, while c is the rate of income.) Assume henceforth the case $\lambda\mu/c < 1$. With $a(w) = R(0)$, Theorem 4.1 continues to apply in this degenerate case to inform us

$$R(w) = a(w) + \int_0^w a(w - y)\, dM(y) = R(0)[1 + M(w)],$$

and since $M(w)$ corresponds to a terminating renewal process

$$\lim_{w\to\infty} M(w) = L/(1 - L) = \frac{\lambda\mu/c}{1 - \lambda\mu/c}$$

$L = \lambda\mu/c$ is presumably from the upper limit of improper density

whence

$$\lim_{w\to\infty} R(w) = \frac{R(0)}{1 - (\lambda\mu/c)}.$$

But $R(\infty) = 1$ (why?), and we obtain

$$R(0) = 1 - \frac{\lambda\mu}{c}.$$

More precise asymptotic relations can be achieved by refining the analysis.

8: *More Elaborate Applications of Renewal Theory*

A. A GENETIC MODEL WITH MUTATION

Consider a finite population of constant size N and label the individuals by $j = 1, \ldots, N$. This comprises the first generation in an evolutionary process subject to certain natural selection effects and mutation pressures. We now delimit the nature and order of the forces governing the process.

Each individual of the population is endowed with a characteristic called "fitness." Loosely speaking, fitness is a measure of the individual's innate relative advantage in contributing offspring to the succeeding generation. Let w_k^1 denote the fitness of the kth individual of the first generation. Determine

$$u_k = \frac{w_k^1}{\sum_{j=1}^{N} w_j^1}, \qquad k = 1, 2, \ldots, N, \tag{8.1}$$

which connotes the relative fitness value of the kth individual. The next generation of progeny is formed by performing N independent random samplings following a multinomial distribution with probability vector (8.1). Thus an offspring carries the fitness value w_k^1 of his parental type and will be selected with probability u_k. Manifestly, individuals of high fitness value compared to the others have concordantly larger relative fitness values and manifestly have greater chance of propagating their own kind.

This multinomial reproduction procedure bears a population of offspring carrying fitness values

$$\tilde{w}^2 = (\tilde{w}_1^2, \tilde{w}_2^2, \ldots, \tilde{w}_N^2). \tag{8.2}$$

(Each of the values $\tilde{w}_i^2$ is, of course, one of the $\{w_k^1\}$. The type of an individual will be identified with his fitness.)

The vector $\tilde{w}^2$ does not yet comprise the mature population of the second generation. We will introduce the possibilities of mutation, so that an offspring can undergo a spontaneous change in fitness value. The precise assumption concerning the effects of the mutation changes is as follows: We suppose that $\{V_i^j;\ i = 1, \ldots, N,\ j = 2, \ldots\}$ is a rectangular array of independent, identically distributed positive random variables, and then let

$$\begin{aligned} w_1^2 &= \tilde{w}_1^2 V_1^2, \\ w_2^2 &= \tilde{w}_2^2 V_2^2, \\ &\vdots \\ w_N^2 &= \tilde{w}_N^2 V_N^2. \end{aligned}$$

The vector $(w_1^2, \ldots, w_N^2)$ represents the fitnesses of mature individuals in the second generation. The above procedure is repeated, sequentially producing successive N-dimensional vectors that depict the evolution of the population through the changes of the fitness vector $w^k = (w_1^k, \ldots, w_N^k)$, and the relative fitness vector $u^k = (u_1^k, \ldots, u_N^k)$ [the superscript indicates the generation number counting from the initial specified population of (8.1)]. The probability law governing the determination of w^{k+1} from w^k and u^k in line with the formation of w^2 from w^1 goes as follows: Sample N independent values from among $w_1^k, w_2^k, \ldots, w_N^k$ with probabilities u_i^k of choosing w_i^k, $i = 1, 2, \ldots, N$. Denote the resulting vector by $\tilde{w}^{k+1} = (\tilde{w}_1^{k+1}, \ldots, \tilde{w}_N^{k+1})$. Mutation changes then transform $\tilde{w}^{k+1}$ to w^{k+1} through multiplication by the positive random variables V_i^{k+1} in the explicit manner

$$w_i^{k+1} = \tilde{w}_i^{k+1} V_i^{k+1}, \qquad i = 1, 2, \ldots, N.$$

Finally, determine the relative fitness vector u^{k+1} by the rule

$$u_i^{k+1} = \frac{w_i^{k+1}}{\sum_{j=1}^{N} w_j^{k+1}}, \qquad i = 1, 2, \ldots, N.$$

The evolutionary process can be realized by the path of the point w^k, $k = 1, 2, \ldots$, traversed in N-dimensional space.

The relative fitness u^k is the projection of the random vector w^k onto the N-dimensional simplex

$$\Delta_N = \{x = (x_1, \ldots, x_N) : x_i \geq 0 \text{ and } x_1 + \cdots + x_N = 1\}.$$

Figure 9 illustrates the projection when $N = 3$. As generations pass, u^k describes the relative fitness point moving about the simplex Δ_N. It is natural to inquire concerning the long-run statistical behavior of u^k.

Define $T(0)$ as the elapsed number of generations (i.e., the smallest $k - 1$) until all components of $\tilde{w}^k$ coincide. Such a generation is called a ***generation of equal components.***

means the generation where
something has taken over
so everything is likely to
be like most fit individual